C^*-algebras associated to quotient maps

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PARS 2014

Acknowledgment

This is joint work with Astrid an Huef and Iain Raeburn.

Outline

- Part 1: Groupoids $R(\psi)$ where $\psi : Y \to X$.

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- Part 2: The C^*-algebras associated to groupoids of the form $R(\psi)$.

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- **Étale Groupoid** A topological groupoid in which the range and source maps are local homeomorphism.

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Example

Take $Y := [0, 1]$ and $X := \{a, b\}$ with topology $\{X, \{a\}, \emptyset\}$. Define $\psi : Y \to X$ by

$$\psi(y) = \begin{cases} a & \text{if } y > 0 \\ b & \text{if } y = 0. \end{cases}$$

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Then ψ is a quotient map and

$$R(\psi) = ((0,1] \times (0,1]) \cup \{(0,0)\}$$

is not a locally compact subset of $[0,1] \times [0,1]$.

Recall that $\psi : Y \to X$ where X is any topological space.

- The space X is not Hausdorff in general but if $R(\psi)$ is locally compact, then X is T_1.

Cartan Groupoids

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Corollary

The groupoid G is Cartan if and only if the topology on G is the relative product topology on $G^{(0)} \times G^{(0)}$.

Part 2: C^*-algebras

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Proof.

Because X is a T_1 space, all representations of $C^*(R(\psi))$ are induced. [C (2007)] □

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  $\delta(A) = 0$ if and only if A is Morita equivalent to $C^*(R(\psi))$ for some ψ.

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Twisted algebras

- We say σ is a 2-cocycle on a groupoid G if $\sigma : G^{(2)} \to \mathbb{T}$ such that

$$\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma).$$

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- We write $C^*(G, \sigma)$ for the twisted C^*-algebra associated to a groupoid G; that is, the completion of $C_c(G, \sigma)$ (where convolution and involution are "twisted" by σ).

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1. $G \cong R(\psi)$ *for some quotient map* ψ*;*
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- NOTE: We reconciled various notions of 'twisted groupoid C^*-algebras'. For LCH principal and Cartan groupoid G (with LHS) and continuous normalised 2-cocylce σ we show

$$C^*(E_\sigma; G) \cong C^*(G, \sigma) = C_r^*(G, \sigma) \cong C^*(\Gamma_\sigma; G)$$

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Thank you.